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The topological properties of magnetic helicity

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The relation of magnetic helicity to the topological structure of field lines is discussed. If space is divided into a collection of flux tubes, magnetic helicity arises from internal structure within a flux tube, such as twist and kinking, and external relations between flux tubes, i.e. linking and knotting. The concepts of twist number and writhing number are introduced from the mathematical-biology literature to describe the contributions to helicity from twist about the axis of a flux tube, and from the structure of the axes themselves.

There exists no absolute measure of the helicity within a subvolume of space if that subvolume is not bounded by a magnetic surface. However, a topologically meaningful and gauge-invariant relative measure of helicity for such volumes is presented here. The time derivative of this relative measure is calculated, which leads to an expression for the flow of topological structure across boundaries.

1. Introduction

Many vector fields in nature are divergence-free; thus, for example, magnetic field lines, vortex lines, and the streamlines of incompressible fluids do not have endpoints. This property allows us to examine such field structures in terms of the topology of closed curves (some complications arise when field lines are ergodic rather than truly closed, as is discussed in Arnol'd 1974). The topology of curves (e.g. White 1969; Rolfsen 1976) is of general scientific interest, and has proved useful in such areas as the study of DNA structure (Fuller 1971, 1978; Crick 1976; Pohl 1980), the description of three-dimensional excitations of chemical and biological media (Winfree & Strogatz 1984), and the study of polymer chains (Frank-Kamenetskii, Lukashin & Vologodski 1975).

Moreau (1961) and Moffatt (1969, 1978, 1981) have shown that a pseudoscalar 'helicity' integral of the form $\int \boldsymbol{X}\cdot\nabla\times\boldsymbol{X}\,\mathrm{d}^3x$ can be associated with the topological properties of the field lines of $\nabla\times\boldsymbol{X}$. For example, the magnetic helicity

$$H=\int \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}^3x, \tag{1}$$

with $\boldsymbol{A}$ the vector potential, measures the linkage of magnetic field lines; for two untwisted closed flux tubes linked once (and with a volume of integration containing both tubes)

$$H=\pm 2\Phi_1\Phi_2, \tag{2}$$

where Φ_1 and Φ_2 measure the magnetic flux of the tubes, and the sign of H depends on the sense of linkage. This result follows from Stokes' theorem, which relates the line integral of $\boldsymbol{A}$ along a closed field line to the total flux linking the line. Furthermore, Arnol'd (1974) has found that helicity integrals can be described mathematically in terms of topological objects such as the Hopf invariant and the Gauss linkage integral (e.g. (17) below).

This paper will investigate in detail the relation between helicity integrals and field structure. For definiteness, we will concentrate on the properties of magnetic helicity, but many of the results presented here are of a general nature. Particular emphasis will be placed on describing a topological measure of the field structure contained in an arbitrary subregion of space. Unfortunately, the integral in equation (1) has topological meaning only if the volume of integration $\mathscr{V}$ is over all space (and the fields vanish at infinity), or, more generally, if the boundary S of $\mathscr{V}$ is a magnetic surface ($\boldsymbol{B}\cdot\hat{\boldsymbol{n}}|_S = 0$). Otherwise, some field lines will cross the boundary and close outside $\mathscr{V}$. The linkage properties of these lines with the other lines inside $\mathscr{V}$ will then be ill-determined, given only information about the field inside $\mathscr{V}$.

An associated problem is that of gauge-invariance: Let $\boldsymbol{A}\rightarrow\boldsymbol{A}+\nabla\xi$. Then from (1) the change in H is

$$\begin{aligned}\Delta H &= \int_{\mathscr{V}} \nabla\xi\cdot\boldsymbol{b}\,\mathrm{d}^3x \\ &= \int_{\mathscr{V}} \nabla\cdot\xi\boldsymbol{B}\,\mathrm{d}^3x \\ &= \oint_S \xi\boldsymbol{B}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S. \end{aligned} \tag{3}$$

Only if S is a magnetic surface will the helicity integral be gauge-invariant. In the last step of (3) it has been assumed that the gauge transformation $\nabla\xi$ is defined throughout a simply connected volume containing $\mathscr{V}$. Otherwise (for example, if $\mathscr{V}$ is a torus and ξ measures toroidal angle), ξ will be multivalued. H will then be gauge-invariant within a magnetic surface with the restriction that the line integrals of $\boldsymbol{A}$ about the holes in the volume be specified (Moreau 1961; Taylor 1981).

Note that, if $\boldsymbol{B}\cdot\hat{\boldsymbol{n}}|_S \neq 0$, we cannot simply choose to work in Coulomb gauge in order to define H; Coulomb gauge is ill-defined inside $\mathscr{V}$ without a knowledge of the outside field. The different divergence-free vector potentials inside $\mathscr{V}$ correspond to Coulomb potentials of fields which have differing structures (and perhaps differing linkage properties) outside $\mathscr{V}$. The gauge-invariance problem is not restricted to magnetic helicity. Although the fluid helicity

$$I = \int_{\mathscr{V}} \boldsymbol{V}\cdot\nabla\times\boldsymbol{V}\,\mathrm{d}^3x \tag{4}$$

is based upon a physically well-defined density $\boldsymbol{V}\cdot\boldsymbol{\omega}$ (where $\boldsymbol{\omega} = \nabla\times\boldsymbol{V}$), it can be related to the topological structure of the vortex field $\boldsymbol{\omega}$ only if adding potential flows to $\boldsymbol{V}$ does not affect I. Again, gauge invariance is ensured only if the boundary surface S has $\boldsymbol{\omega}\cdot\hat{\boldsymbol{n}}|_S = 0$.

In spite of the above restrictions, it seems plausible that there should be some well-defined measure of the linkage due to the twisting and tangling of field lines in a region of space not bounded by a magnetic surface, just as one can describe the amount of supercoiling of a length of telephone cord, or of a segment of DNA (Fuller 1978). In §3 we will show that such a measure does exist: for any simply connected volume $\mathscr{V}$, the *difference* in helicities (integrated over all space) of any two field configurations that differ only inside $\mathscr{V}$ is independent of the structure of the fields external to $\mathscr{V}$. Thus, the volume $\mathscr{V}$'s contribution to the overall helicity of a field has a well-defined relative measure. A particularly useful reference field inside $\mathscr{V}$ is the potential field, as it is completely determined by $\boldsymbol{B}\cdot\hat{\boldsymbol{n}}|_S$.

The magnetic helicity is a constant of the motion in ideal magnetohydrodynamics

(MHD) (Elsasser 1956; Woltjer 1958), as the field lines (and their linkage properties) are frozen into the fluid as it moves. Similarly, the fluid helicity (Moffatt 1969) is a constant of the motion in ideal hydrodynamics when all external forces are potential. As Carter (1979) has noted, fluid helicity as a conserved quantity provides a volume-integral counterpart to surface integrals of vorticity and line integrals of circulation. The conservation equation for the magnetic-helicity density h_0 is readily derived from the homogeneous Maxwell equations, and can be written in the form

$$\frac{\partial h_0}{\partial t}+c\boldsymbol{\nabla}\cdot\boldsymbol{h} = -2c\boldsymbol{E}\cdot\boldsymbol{B}, \tag{5}$$

where $h_0 = \boldsymbol{A}\cdot\boldsymbol{B}$ and $\boldsymbol{h} = \boldsymbol{E}\times\boldsymbol{A}+\phi\boldsymbol{B}$. Here ϕ is the scalar potential. The source term $\boldsymbol{E}\cdot\boldsymbol{B}$ vanishes in ideal MHD. Note that each term in the equation is a pseudoscalar. This equation is fully relativistic, and can be expressed in 4-vector language (Carter 1978).

Two important classes of electric fields conserve helicity. First, if $\boldsymbol{E}$ can be derived from a potential, $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$, then $\boldsymbol{\nabla}\cdot\boldsymbol{h} = -2\boldsymbol{E}\cdot\boldsymbol{B}$. Thus $\partial h_0/\partial t = 0$, as expected from the induction equation as applied to electrostatic fields:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla}\times\boldsymbol{E} = 0. \tag{6}$$

As a second example, in ideal MHD $\boldsymbol{E} = \boldsymbol{B}\times \boldsymbol{V}/c$, so that

$$c\boldsymbol{h} = h_0\boldsymbol{V}-(\boldsymbol{A}\cdot\boldsymbol{V})\boldsymbol{B}; \tag{7}$$

this leads to conservation of the total helicity contained in comoving volumes bounded by magnetic surfaces also moving with the fluid (Moffatt 1969). These results bear close analogy to Newcomb's (1958) analysis of field-line motion. First, the sufficient conservation condition for H,

$$\boldsymbol{\nabla}\times\left(\boldsymbol{E}+\frac{1}{c}\boldsymbol{V}\times\boldsymbol{B}\right) = 0, \tag{8}$$

is equivalent to Newcomb's condition for the existence of flux-preserving velocity fields. For example, when H arises from the linkage of flux loops, H-conservation follows from preservation of the flux linking each field line in the loops. Also, the stronger and relativistically invariant condition, $\boldsymbol{E}\cdot\boldsymbol{B} = 0$, holds true if and only if there exist relativistic generalizations of field lines, i.e. surfaces in space–time that (in any reference frame) are traced out by moving magnetic lines.

Magnetic helicity may not be conserved when finite resistivity is present. Assuming a linear Ohm's law,

$$\boldsymbol{E}-\frac{1}{c}\boldsymbol{B}\times\boldsymbol{V} = \eta\boldsymbol{J}, \tag{9}$$

the helicity-dissipation rate is

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -2c\int_{\mathscr{V}}\eta\boldsymbol{J}\cdot\boldsymbol{B}\,\mathrm{d}^3x. \tag{10}$$

This dissipation rate may often be small, however: Taylor (1974, 1981) has conjectured that helicity should be approximately conserved on ideal or reconnection time-scales in high-magnetic-Reynolds-number plasmas. Furthermore, Berger (1984) has obtained strict limits on helicity decay in an isolated plasma. These limits tend to support the Taylor conjecture.

The plan of this paper is as follows. In §2 we will relate magnetic helicity to the morphology of closed field structures. Helicity will be described in terms of the internal structure of a flux tube, and the external relations between flux tubes. Also, the concepts of twist number and writhing number (Fuller 1971, 1978) will be introduced. In §3 the helicity of open field configurations not bounded by magnetic surfaces will be discussed, and a measure of the propagation of topological structure across open boundaries will be given in §4. Conclusions will be presented in §5.

2. The helicity of closed field structures

In analysing the morphology of a field, it is often useful to separate space into regions bounded by magnetic surfaces. The magnetic helicity of the field can then be decomposed into a sum of internal helicities corresponding to structure inside each region, and external helicities due to interlinkages among the regions. This decomposition is conserved in ideal MHD. If the field structure is relatively simple, the separatrices of the field provide a natural choice for the magnetic surfaces, as they separate space into cells consisting of topologically equivalent field lines (i.e. the lines within a given cell have identical linkage properties). If the field were ergodic within a finite volume, however, the concept of separatrix surface would break down, and it would then become necessary to integrate the helicity over the entire ergodic region.

The separatrix structure can also become complicated if the field resides in a very large or unbounded region. Suppose that the field structure looks simple locally, but that any two neighbouring field lines would be seen to diverge if they were followed sufficiently far. In that case, the cells defined by the separatrices would each carry infinitesimal flux. This situation may be dealt with by assuming a simple form for the faraway field – the fewer branchings included in the description of the field, the simpler the separatrix structure. In §3 it will be shown that, for the purpose of comparing different local field configurations, it does not matter which external field is chosen.

Internal helicity can easily be computed for a volume consisting of nested toroidal magnetic surfaces. For a particular surface, let Ψ_P be the poloidal flux threading the hole of the toroidal surface, and let Ψ_T be the toroidal flux contained within the surface. For an infinitesimal annular volume containing flux $(d\Psi_T, -d\Psi_P)$, the linkage helicity with the fields outside (counted once) is $\Psi_P \, d\Psi_T$ (i.e. the toroidal flux within the annulus $d\Psi_T$ links the poloidal flux Ψ_P as in (2)). Similarly, the linkage helicity with the fields within is $-\Psi_T \, d\Psi_P$. Thus the helicity of the annular volume is

$$dH = \Psi_P \, d\Psi_T - \Psi_T \, d\Psi_P; \tag{11}$$

this result was obtained analytically by Kruskal & Kulsrud (1958).

After integrating the first term by parts, we obtain

$$H = 2\int_0^{\Phi} T\Psi_T \, d\Psi_T, \tag{12}$$

where Φ is the total toroidal flux, and $T = -d\Psi_P/d\Psi_T$. T can be any real number, and represents the number of times a field line winds around the torus in the poloidal direction (the short way around) for one circuit in the toroidal direction. For a uniformly twisted torus with T twists,

$$H = T\Phi^2. \tag{13}$$

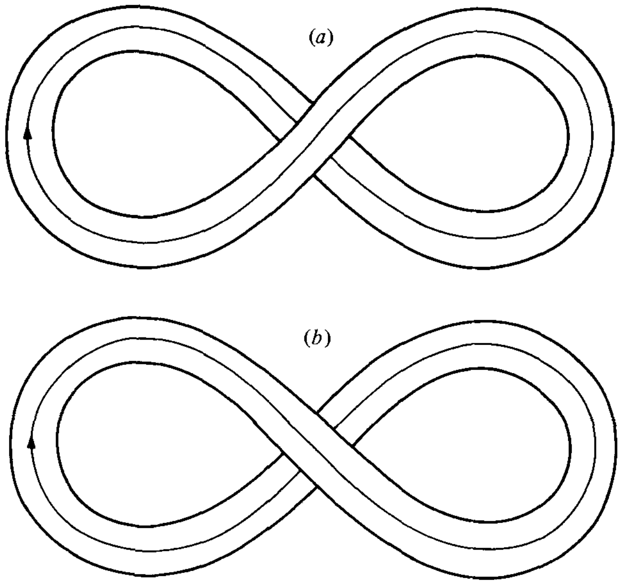

FIGURE 1. Figure-of-8 shapes. A typical field line is shown. (*a*) $H_K = +\Phi^2$; (*b*) $H_K = -\Phi^2$.

Note that a torus with $T = \pm 1$ may be distorted, or kinked, into a figure-of-8 configuration which appears untwisted (see figure 1). The internal helicity of the torus will in general be manifest as some combination of twist and kink helicity (plus, perhaps, contributions from a stochastic component of the field).

A simple formula may be derived for the helicity of an arbitrarily twisted, kinked and knotted flux tube. We express H as the sum of twist helicity and of knot (and kink) helicity,

$$H = H_T + H_K. \tag{14}$$

This decomposition is not topologically invariant, as it separates twists and kinks, which can convert into one another. To fix the zero of H_T, we adopt the convention that, for an 'untwisted' knotted or kinked tube, a field line can be drawn on top of the tube along its entire length as viewed in a plane projection (figure 2*a*). This line sets $\theta = 0$ for a poloidal coordinate system on the tube; the winding number of a twisted tube can be defined with respect to this system. An alternative definition of H_T and H_K which does not involve a particular plane projection will be given below.

At each crossover exhibited by the knot, we reconnect (without changing H_T or H_K) the magnetic lines on either side of the crossover to create a figure of 8 (figures 2*b, c*). A crossover is called positive or negative according to the sign of H for the corresponding figure of 8. Then

$$H_K = \Phi^2(N_+ - N_-), \tag{15}$$

i.e. the flux squared times the number of positive crossovers minus the number of negative crossovers. Note that the result for the trefoil knot with a minimum of crossovers ($H_K = \pm 3\Phi^2$) differs from Moffatt's result (1978, p. 15); kink helicity was not properly accounted for in his calculation. A collection of linked and knotted flux ropes may be dealt with in a similar fashion. In this case, one must reconnect each individual rope to itself (because different ropes may have different fluxes) to finally

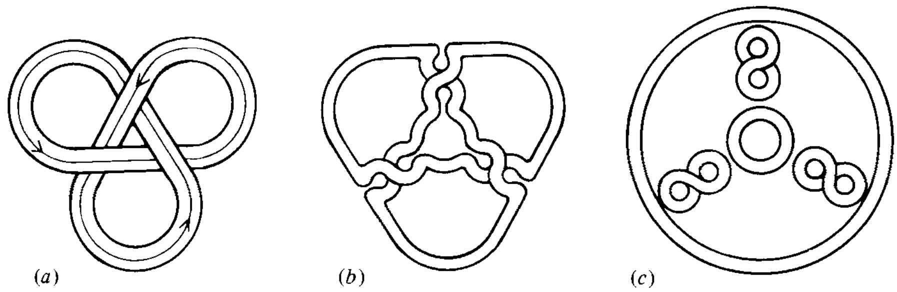

FIGURE 2. A trefoil knot with $H_K = -3\Phi^2$.

obtain a collection of interlinked simple loops and figure of eights. Equation (15) still holds, with Φ^2 replaced by a product of the two fluxes present at a particular crossover.

The external helicity of a knot we define as equalling H_K if the knot exhibits a minimum of crossovers. For two linked flux tubes, the external helicity is $2L_{12}\Phi_1\Phi_2$, where L_{12} gives the number of linkages (Moffatt 1969); this is a generalization of (2). Note, however, that, as L_{12} increases, the tubes must become more and more kinked, so that internal helicity cannot be ignored in computing the total helicity. There is one situation where internal helicity can indeed be neglected: one may compute the helicity of a configuration by approximating the field as consisting of a large number N of closed flux elements, each containing a small flux $\delta\Phi$. In this case,

$$H = \sum_{i=1}^{N}\sum_{j=1}^{N} L_{ij}(\delta\Phi)^2 + \sum_{i=1}^{N} H_{i,\,\mathrm{int}}. \tag{16}$$

Note that $(\delta\Phi)^2$ and the internal helicities $H_{i,\,\mathrm{int}}$ vary as N^{-2}. As $N \to \infty$ the interlinkage sum will reach a finite limit since it contains N^2 terms, but the internal helicity sum vanishes as N^{-1}.

The decomposition into H_T and H_K depends on the angle of projection employed to find the crossovers. This deficiency can be removed by introducing the concepts of twist number Tw and writhing number Wr (Fuller 1971, 1978). Let $\boldsymbol{X}(s)$ be a closed curve, where s parametrizes length along the curve. Also let $\boldsymbol{U}(s)$ be a vector perpendicular to $\boldsymbol{X}(s)$; the 'tip' of $\boldsymbol{U}(s)$ defines a second closed curve $\boldsymbol{Y}(s)$, and the surface between the two curves can be visualized as a ribbon. In the DNA application, $\boldsymbol{X}(s)$ can be taken to be the axis of the molecule, and the outer curve $\boldsymbol{Y}(s)$ to be one of the two nucleotide chains. Similarly, we may take $\boldsymbol{X}(s)$ to be the central axis of a magnetic flux rope, and $\boldsymbol{Y}(s)$ to be a field line winding its way along some toroidal surface within the flux rope.

The linking number L_{XY} of the two curves can be found from the Gauss linkage formula:

$$L_{XY} = -\frac{1}{4\pi}\oint \mathrm{d}s \oint \mathrm{d}s'\, \frac{\mathrm{d}\boldsymbol{X}(s)}{\mathrm{d}s}\cdot\frac{\boldsymbol{r}}{r^3}\times\frac{\mathrm{d}\boldsymbol{Y}(s')}{\mathrm{d}s'}, \tag{17}$$

where $\boldsymbol{r} = \boldsymbol{X}(s) - \boldsymbol{Y}(s')$. When integrated over all space, the helicity integral can be expressed in a similar form (Moffatt 1969, 1981; Arnol'd 1974). In Coulomb gauge the vector potential $\boldsymbol{A}$ is given by

$$A(x) = \frac{1}{c}\int \mathrm{d}^3x'\, \frac{\boldsymbol{J}(\boldsymbol{x}')}{r}, \tag{18}$$

where $\boldsymbol{r} = \boldsymbol{x}-\boldsymbol{x}'$. Using $\boldsymbol{J} = (c/4\pi)\nabla\times\boldsymbol{B}$ and integrating by parts, we find that

$$\boldsymbol{A}(\boldsymbol{x}) = -\frac{1}{4\pi}\int \mathrm{d}^3x'\,\frac{\boldsymbol{r}}{r^3}\times\boldsymbol{B}(\boldsymbol{x}'), \tag{19}$$

which gives

$$H = -\frac{1}{4\pi}\int \mathrm{d}^3x \int \mathrm{d}^3x'\boldsymbol{B}(\boldsymbol{x})\cdot\frac{\boldsymbol{r}}{r^3}\times\boldsymbol{B}(\boldsymbol{x}'). \tag{20}$$

The twist number is given by

$$Tw = \frac{1}{2\pi}\oint \mathrm{d}s\,\frac{\mathrm{d}\hat{\boldsymbol{U}}(s)}{\mathrm{d}s}\cdot\hat{\boldsymbol{U}}(s)\times\boldsymbol{X}(s), \tag{21}$$

with $\hat{\boldsymbol{U}} = \boldsymbol{U}/|\boldsymbol{U}|$ (in general, (21) will not work for a field line $\boldsymbol{Y}$ that travels perpendicular or backwards with respect to $\boldsymbol{X}$, because then $\hat{\boldsymbol{U}}(s)$ would not be single-valued. However, Tw can still be defined for ill-behaved $\boldsymbol{Y}$-curves via (23) below.) Finally, the writhing number is the Gauss linkage integral applied to the axis:

$$Wr = -\frac{1}{4\pi}\oint \mathrm{d}s \oint \mathrm{d}s'\,\frac{\mathrm{d}\boldsymbol{X}(s)}{\mathrm{d}s}\cdot\frac{\boldsymbol{r}}{r^3}\times\frac{\mathrm{d}\boldsymbol{X}(s')}{\mathrm{d}s'}, \tag{22}$$

where $\boldsymbol{r} = \boldsymbol{X}(s)-\boldsymbol{X}(s')$. Wr can be shown to equal the signed sum of crossovers (as in (15)) exhibited by the axis curve, averaged over all projection angles (Fuller 1978).

White (1969) has proved that

$$L_{XY} = Tw + Wr \tag{23}$$

(an earlier, more restricted version of this theorem was found by Călugăreanu 1959). Of these quantities, only Tw can be defined for a subsection of a ribbon, as L_{XY} and Wr are given by double integrals. On the other hand, only L_{XY} is topologically invariant, i.e. unchanged by deformations of $\boldsymbol{X}$ and $\boldsymbol{Y}$ that do not let the two curves cross each other. Furthermore, the writhing number has the unique property of depending only upon the geometry of the axis curve $\boldsymbol{X}$.

Given the basic interpretation of magnetic helicity as arising from the linkage of field lines, the similarity between (14) and (23) is not surprising. H_{T} and H_{K}, previously calculated using one projection angle as in figure 2, may alternatively be averaged over all projection angles through the use of (23). To see this explicitly, consider a small bundle of field lines with net flux $\delta\Psi_{\mathrm{T}}$ in the neighbourhood of a particular line $\boldsymbol{Y}$ (assume for simplicity that $\boldsymbol{Y}$ is a closed curve – the case of curves with irrational twist ergodically covering a flux surface can be dealt with via a limiting process (Arnol'd 1974; Fuller 1978). The total linkage helicity between the flux element $\delta\Psi_{\mathrm{T}}$ and the flux Ψ_{T} interior to $\boldsymbol{Y}$ is

$$\delta H = 2L_{XY}\,\Psi_{\mathrm{T}}\,\delta\Psi_{\mathrm{T}}. \tag{24}$$

Employing (23) and integrating, we obtain expressions for H_{T} and H_{K} in terms of the twist number and the writhing number:

$$H_{\mathrm{T}} = \int_0^{\Phi^2} Tw\,\mathrm{d}\,\Psi_{\mathrm{T}}^2, \tag{25}$$

$$H_{\mathrm{K}} = Wr\,\Phi^2. \tag{26}$$

If the flux tube is relatively flat, so that one sees the same signed number of crossovers (15) from nearly all projection angles, then the two definitions for H_{T} and H_{K} given in this section will be approximately equivalent.

3. The helicity of open field structures

The linking and writhing numbers of a ribbon are well-defined only if the ribbon is closed. However, Fuller (1978) demonstrated that the linking number of an open segment of a ribbon can be defined relative to a reference segment with the same endpoints. This result has application to the study of compact protein structures, called nucleosomes, which act as spools around which the DNA of higher organisms is wrapped (Crick 1976; Worcel, Strogatz & Riley 1981). The difference in linking numbers between a closed ribbon containing a nucleosome segment, and the same ribbon with a reference segment substituted in place of the nucleosome, is only weakly dependent on the exterior ribbon structure (totally independent if the exterior ribbon remains outside of some simply connected volume containing the nucleosome). We will prove an analogous result for the topology of continuous fields.

Let space $\mathscr{V}$ be divided into two simply connected regions $\mathscr{V}_a$ and $\mathscr{V}_b$ separated by a boundary surface S (in general $\mathscr{V}_a$ and $\mathscr{V}_B$ could each be unions of separated simply connected components). A divergence-free field $\boldsymbol{B}$ in $\mathscr{V}$ will be denoted by an ordered pair, for example

$$\boldsymbol{B} \equiv (\boldsymbol{B}_a, \boldsymbol{B}_b), \tag{27}$$

whose value at a point $\boldsymbol{x}$ is

$$\boldsymbol{B}(\boldsymbol{x}) = \begin{cases} \boldsymbol{B}_a(\boldsymbol{x}) & \text{if} \quad \boldsymbol{x} \in \mathscr{V}_a, \\ \boldsymbol{B}_b(\boldsymbol{x}) & \text{if} \quad \boldsymbol{x} \in \mathscr{V}_b. \end{cases} \tag{28}$$

To ensure $\nabla \cdot \boldsymbol{B} = 0$, we must require that

$$\boldsymbol{B}_a \cdot \hat{\boldsymbol{n}}|_S = \boldsymbol{B}_b \cdot \hat{\boldsymbol{n}}|_S, \tag{29}$$

where

$$\hat{\boldsymbol{n}} \equiv \hat{\boldsymbol{n}}_a = -\hat{\boldsymbol{n}}_b \tag{30}$$

is a unit normal pointing away from $\mathscr{V}_a$.

Let $H(\boldsymbol{B}) = H(\boldsymbol{B}_a, \boldsymbol{B}_b)$ denote the helicity of the field described by (28) integrated over the entire volume $\mathscr{V}$. We consider only fields whose source currents exist in a finite region of space. This guarantees the gauge invariance of $H(\boldsymbol{B})$, as the 'surface at infinity' will be a magnetic surface.

Consider two fields $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ that differ only in $\mathscr{V}_a$. We wish to show that

$$\Delta H \equiv H(\boldsymbol{B}_1) - H(\boldsymbol{B}_2), \tag{31}$$

is independent of their common extension into $\mathscr{V}_b$ (figure 3). Using the terminology of (27) and (28), $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ can be written

$$\boldsymbol{B}_1 = (\boldsymbol{B}_{1a}, \boldsymbol{B}_b), \tag{32}$$

$$\boldsymbol{B}_2 = (\boldsymbol{B}_{2a}, \boldsymbol{B}_b). \tag{33}$$

The theorem to be proven states that ΔH can be calculated without any knowledge of $\boldsymbol{B}_b$. In other words,

$$H(\boldsymbol{B}_{1a}, \boldsymbol{B}_b) - H(\boldsymbol{B}_{2a}, \boldsymbol{B}_b) = H(\boldsymbol{B}_{1a}, \boldsymbol{B}'_b) - H(\boldsymbol{B}_{2a}, \boldsymbol{B}'_b), \tag{34}$$

for any fields $\boldsymbol{B}_b$ and $\boldsymbol{B}'_b$ that satisfy the given boundary conditions (29) at S.

Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be vector potentials for $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$. From (1)

$$\Delta H = \int_{\mathscr{V}} (\boldsymbol{A}_1 \cdot \boldsymbol{B}_1 - \boldsymbol{A}_2 \cdot \boldsymbol{B}_2)\, \mathrm{d}^3x \tag{35}$$

$$= \int_{\mathscr{V}} (\boldsymbol{A}_1 - \boldsymbol{A}_2) \cdot (\boldsymbol{B}_1 + \boldsymbol{B}_2)\, \mathrm{d}^3x + \int_{\mathscr{V}} (\boldsymbol{A}_2 \cdot \boldsymbol{B}_1 - \boldsymbol{A}_1 \cdot \boldsymbol{B}_2)\, \mathrm{d}^3x. \tag{36}$$

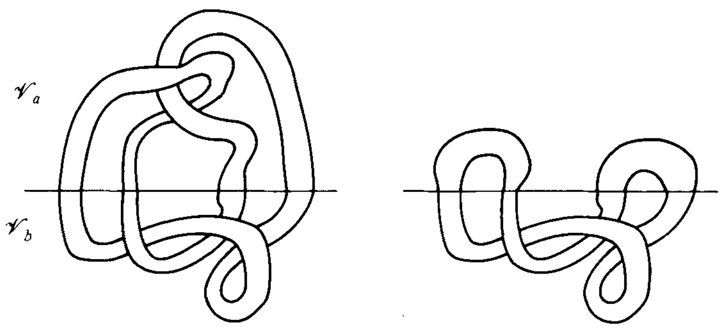

FIGURE 3. The difference in total helicity of these two configurations is independent of the field in $\mathscr{V}_b$.

By an integration by parts, the second integral vanishes. Also note that $\nabla \times \boldsymbol{A}_1 = \nabla \times \boldsymbol{A}_2$ inside $\mathscr{V}_b$. Hence, inside $\mathscr{V}_b$, $\boldsymbol{A}_1 - \boldsymbol{A}_2 = \nabla\chi$ for some scalar χ. Separating the first integral into contributions from $\mathscr{V}_a$ and $\mathscr{V}_b$,

$$\Delta H = \int_{\mathscr{V}_a} (\boldsymbol{A}_1 - \boldsymbol{A}_2)\cdot(\boldsymbol{B}_1 + \boldsymbol{B}_2)\,\mathrm{d}^3x + \int_{\mathscr{V}_b} \nabla\chi\cdot(\boldsymbol{B}_1 + \boldsymbol{B}_2)\,\mathrm{d}^3x, \tag{37}$$

$$= \int_{\mathscr{V}_a} (\boldsymbol{A}_1 - \boldsymbol{A}_2)\cdot(\boldsymbol{B}_{1a} + \boldsymbol{B}_{2a})\,\mathrm{d}^3x - \oint_S \chi(\boldsymbol{B}_{1a} + \boldsymbol{B}_{2a})\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S. \tag{38}$$

Note that the assumption of simply connected volumes ensures that χ will be single-valued.

From (19), if Coulomb gauge is used for $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, then

$$\boldsymbol{A}_1(\boldsymbol{x}) - \boldsymbol{A}_2(\boldsymbol{x}) = -\frac{1}{4\pi}\int_{\mathscr{V}_a} \frac{\boldsymbol{r}}{r^3} \times (\boldsymbol{B}_{1a}(\boldsymbol{x}') - \boldsymbol{B}_{2a}(\boldsymbol{x}')\,\mathrm{d}^3x'. \tag{39}$$

This shows that in Coulomb gauge $\boldsymbol{A}_1 - \boldsymbol{A}_2$ and χ only depend on the fields inside $\mathscr{V}_a$. Hence (38) proves the assertion that ΔH is independent of $\boldsymbol{B}_b$. One may check that (38) is gauge-invariant: if $\nabla\xi_1$ is added to $\boldsymbol{A}_1$, and $\nabla\xi_2$ is added to $\boldsymbol{A}_2$, then the change in ΔH is

$$\delta\Delta H = \int_{\mathscr{V}_a} (\nabla\xi_1 - \nabla\xi_2)\cdot(\boldsymbol{B}_{1a} + \boldsymbol{B}_{2a})\,\mathrm{d}^3x - \oint_S (\xi_1 - \xi_2)/(\boldsymbol{B}_{1a} + \boldsymbol{B}_{2a})\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S \tag{40}$$

$$= 0. \tag{41}$$

This theorem can be extended to allow $\mathscr{V}$ to be a subvolume of space bounded by a magnetic surface. In this case, the helicity integrated over $\mathscr{V}$ might depend on fields external to $\mathscr{V}$. For example, $\mathscr{V}$ could be a torus, and $\mathscr{V}_a$, $\mathscr{V}_b$ could be created by cutting $\mathscr{V}$ at toroidal angles 0 and π. Any external fields $\boldsymbol{B}_{\text{ext}}$ linking $\mathscr{V}$ would then contribute to the helicity integral. We can express the theorem when $\mathscr{V}$ is a subvolume in a manner similar to (34):

$$H(\boldsymbol{B}_{1a}, \boldsymbol{B}_b, \boldsymbol{B}_{\text{ext}}) - H(\boldsymbol{B}_{2a}, \boldsymbol{B}_b, \boldsymbol{B}_{\text{ext}}) = H(\boldsymbol{B}_{1a}, \boldsymbol{B}'_b, \boldsymbol{B}'_{\text{ext}}) - H(\boldsymbol{B}_{2a}, \boldsymbol{B}'_b, \boldsymbol{B}'_{\text{ext}}), \tag{42}$$

where $\boldsymbol{B}'_{\text{ext}}$ need not equal $\boldsymbol{B}_{\text{ext}}$. By lumping together $\mathscr{V}_b$ and the space external to $\mathscr{V}$ into one region $\mathscr{V}_c$, so that space can again be divided into two regions, i.e. $\mathscr{V}_a$ and $\mathscr{V}_c$, we find that (42) follows directly from (34).

Suppose we wish to examine the helicity of the fields contained within $\mathscr{V}_a$. Because only differences in helicity are meaningful, a reference field will be needed. The

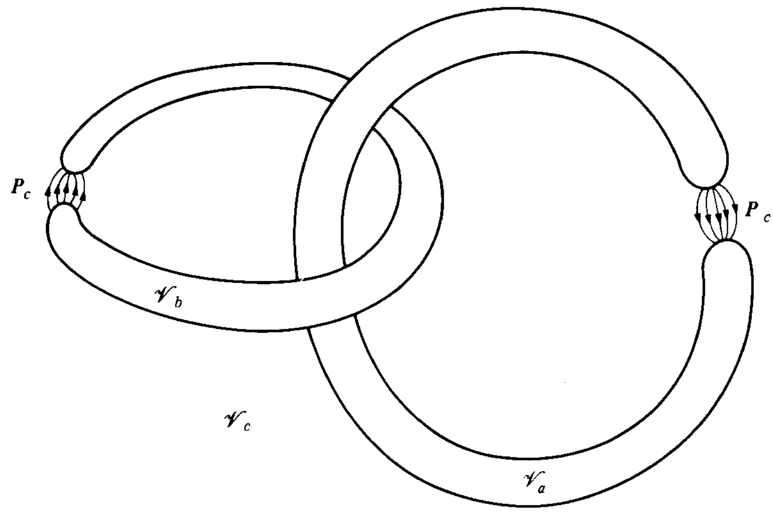

FIGURE 4. Two simply connected surfaces divide space into $\mathscr{V}_a$, $\mathscr{V}_b$ and $\mathscr{V}_c$. The field lines of $\boldsymbol{P}_c$ are shown. For the boundary conditions shown, $H(\boldsymbol{P}_a, \boldsymbol{P}_b, \boldsymbol{P}_c)$ is non-zero.

potential field in $\mathscr{V}_a$, $\boldsymbol{P}_a(\nabla \times \boldsymbol{P}_a = 0)$ is an especially useful reference field because it is completely determined by the boundary conditions $\boldsymbol{B}\cdot\hat{\boldsymbol{n}}|_S$. Furthermore, $\boldsymbol{P}_a$ is the minimum energy field for these boundary conditions. Let the 'relative helicity of $\mathscr{V}_a$', $H_{\mathrm{R}}(\mathscr{V}_a)$, be defined by

$$H_{\mathrm{R}}(\mathscr{V}_a) \equiv H(\boldsymbol{B}_a, \boldsymbol{B}'_b) - H(\boldsymbol{P}_a, \boldsymbol{B}'_b), \tag{43}$$

where, by the theorem just proved, $\boldsymbol{B}'_b$ is arbitrary. Similarly, let

$$H_{\mathrm{R}}(\mathscr{V}_b) \equiv H(\boldsymbol{B}'_a, \boldsymbol{B}_b) - H(\boldsymbol{B}'_a, \boldsymbol{P}_b), \tag{44}$$

with $\boldsymbol{B}'_a$ arbitrary and $\boldsymbol{P}_b$ the potential field inside $\mathscr{V}_b$.

The total helicity inside $\mathscr{V}$ can be decomposed into contributions from the two relative helicities, plus a term due to the potential fields. Choose $\boldsymbol{B}'_a = \boldsymbol{P}_a$ and $\boldsymbol{B}'_b = \boldsymbol{B}_b$ in (43) and (44). Summing the two equations results in

$$H(\boldsymbol{B}_a, \boldsymbol{B}_b) = H_{\mathrm{R}}(\mathscr{V}_a) + H_{\mathrm{R}}(\mathscr{V}_b) + H(\boldsymbol{P}_a, \boldsymbol{P}_b). \tag{45}$$

The last term only depends on the shape of the boundary S, and the distribution of $\boldsymbol{B}\cdot\hat{\boldsymbol{n}}|_S$.

This addition law for relative helicities can be generalized to the case where space is divided into N simply connected volumes $\mathscr{V}_i$, $i = 1, \ldots, n$. Write the magnetic field in a form similar to (27), $\boldsymbol{B} = (\boldsymbol{B}_1, \ldots, \boldsymbol{B}_N)$, and let $\boldsymbol{P}_i$ be the current-free field determined by the normal field component at the boundary of $\mathscr{V}_i$. Then, by choosing

$$H_{\mathrm{R}}(\mathscr{V}_i) = H(\boldsymbol{P}_1, \ldots, \boldsymbol{P}_{i-1}, \boldsymbol{B}_i, \boldsymbol{B}_{i+1}, \ldots, \boldsymbol{B}_N) - H(\boldsymbol{P}_1, \ldots, \boldsymbol{P}_{i-1}, \boldsymbol{P}_i, \boldsymbol{B}_{i+1}, \ldots, \boldsymbol{B}_N), \tag{46}$$

one finds

$$H(\boldsymbol{B}) = \sum_{i=1}^{N} H_{\mathrm{R}}(\mathscr{V}_i) + H(\boldsymbol{P}_1, \ldots, \boldsymbol{P}_N). \tag{47}$$

If the last terms in (45) or (47) are non-zero, then the minimal fields determined by the normal components at the boundaries have net linkage. For example, consider the configuration shown in figure 4. Here $\mathscr{V}_a$ and $\mathscr{V}_b$ are horseshoe-shaped volumes, and $\mathscr{V}_c$ contains the space external to the horseshoes. If $\boldsymbol{B}\cdot\hat{\boldsymbol{n}}|_S$ is non-zero only at

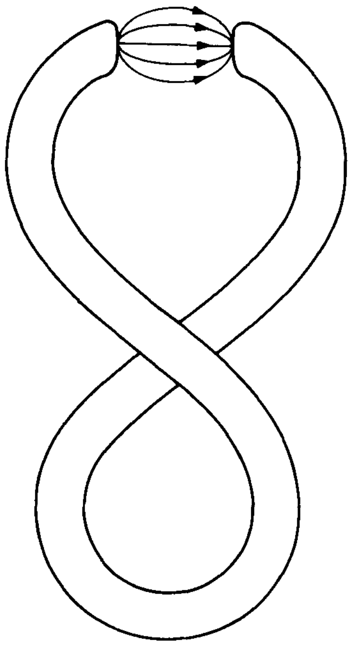

FIGURE 5. A single simply connected surface that can have non-zero $H(\boldsymbol{P}_a, \boldsymbol{P}_b)$.

the ends of the horseshoes, then $\boldsymbol{P}_c$ will be as shown, while the field lines of $\boldsymbol{P}_a$ and $\boldsymbol{P}_b$ (not shown) travel inside $\mathscr{V}_a$ and $\mathscr{V}_b$ more or less parallel to the axes of the horseshoes. The total magnetic field can be treated to a very good approximation as two linked flux tubes with zero twist helicity (there is no toroidal current within $\mathscr{V}_a$ and $\mathscr{V}_b$ to generate poloidal field components). Thus the helicity $H(\boldsymbol{P}_a, \boldsymbol{P}_b, \boldsymbol{P}_c)$ is given by (2).

Figure 5 provides an example of a geometry with only one boundary surface where the potential fields have non-zero helicity. In this configuration, $\mathscr{V}_a$ consists of a horseshoe-shaped volume which has been given one kink. Again, the field pierces the boundary of $\mathscr{V}_a$ only at the ends. Because the interior of $\mathscr{V}_a$ has zero current, $H(\boldsymbol{P}_a, \boldsymbol{P}_b)$ equals the kink helicity H_{K} of a figure-of-8 shape (figure 1). For general fields that satisfy the boundary conditions of figures 4 or 5, the potential field helicity can be subtracted from the total helicity to give us the sum of the relative helicities. The relative helicities thus contain information about the helicity generated by currents within the volumes (for example, the twist helicity H_{T} inside the volumes).

When the boundaries between the $\mathscr{V}_i$ are parallel planes or concentric spheres, the potential-field helicity vanishes identically. For simplicity, consider the case where there is only one boundary surface, and calculate $H(\boldsymbol{P}_a, \boldsymbol{P}_b)$ explicitly. We may write $\boldsymbol{P}_a = \nabla\psi_a$ and $\boldsymbol{P}_b = \nabla\psi_b$, where ψ_a, ψ_b are solutions to the Laplace equation with Neumann boundary conditions. Denote the Coulomb vector potential associated with $(\boldsymbol{P}_a, \boldsymbol{P}_b)$ by $\boldsymbol{A}_{\mathrm{P}}$. Then (using $\nabla\cdot\boldsymbol{A}_{\mathrm{P}} = 0$)

$$H(\boldsymbol{P}_a, \boldsymbol{P}_b) = \oint_S (\psi_a - \psi_b)\, \boldsymbol{A}_{\mathrm{P}}\cdot\hat{\boldsymbol{n}}\, \mathrm{d}S. \tag{48}$$

Note that the source currents for $(\boldsymbol{P}_a, \boldsymbol{P}_b)$ exist as a current sheet on S. For a current sheet confined to a plane (say the (x, y)-plane) we have $J_z = 0$. By (18) this implies that $\boldsymbol{A}_{\mathrm{P}}\cdot\hat{\boldsymbol{n}}|_S = 0$. For a spherical current sheet, let us evaluate $\boldsymbol{A}_{\mathrm{P}}\cdot\hat{\boldsymbol{n}}|_S$ at the north pole ($\theta = 0$). If $\mathscr{J}(\theta, \phi)$ is the current density on the sphere, R the radius, and $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ at $\theta = 0$, then from (18)

$$\boldsymbol{A}_{\mathrm{P}}(\theta = 0)\cdot\hat{\boldsymbol{n}}|_S = \int_0^{\pi} \sin\theta \,\mathrm{d}\theta \,(2R\sin\tfrac{1}{2}\theta)^{-1} \int_0^{2\pi} \mathrm{d}\phi \,\mathscr{J}_z(\theta, \phi). \tag{49}$$

However, the integral of $\mathscr{J}_z$ over azimuthal angle ϕ must vanish; otherwise $\mathscr{J}$ would have non-zero divergence. As the point $\theta = 0$ has no special physical significance, we may say in general that

$$\boldsymbol{A}_{\mathrm{P}}\cdot\hat{\boldsymbol{n}}|_S = 0, \tag{50}$$

when S is a sphere or a plane. Therefore, when $\mathscr{V}$ contains all space and S is a plane or a sphere, $H(\boldsymbol{P}_a, \boldsymbol{P}_b) = 0$. This makes the relative-helicity formulas especially simple:

$$H_{\mathrm{R}}(\boldsymbol{B}_a) = H(\boldsymbol{B}_a, \boldsymbol{P}_b), \tag{51}$$

$$H_{\mathrm{R}}(\boldsymbol{B}_b) = H(\boldsymbol{P}_a, \boldsymbol{B}_b). \tag{52}$$

Also, in this case the sum of the relative helicities does equal the total helicity.

In general, (50) holds true for each boundary when space is divided into subvolumes by parallel planes or concentric spheres. In such geometries $H(\boldsymbol{P}_1, \ldots, \boldsymbol{P}_N) = 0$, and the addition law is

$$H(\boldsymbol{B}) = \sum_{i=1}^{N} H_{\mathrm{R}}(\mathscr{V}_i). \tag{53}$$

4. The flow of helicity across boundaries

The time derivative of $H_{\mathrm{R}}(\mathscr{V}_a)$ provides a gauge-invariant measure of helicity flow across the boundary S, as well as a measure of the internal helicity dissipation inside $\mathscr{V}_a$. By choosing $\boldsymbol{B}'_b = \boldsymbol{P}_b$ in (43),

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{\mathrm{R}}(\mathscr{V}_a) = \frac{\mathrm{d}}{\mathrm{d}t} H(\boldsymbol{B}_a, \boldsymbol{P}_b) - \frac{\mathrm{d}}{\mathrm{d}t} H(\boldsymbol{P}_a, \boldsymbol{P}_b). \tag{54}$$

From (5), $\mathrm{d}H/\mathrm{d}t = -2c\int_{\mathscr{V}} \boldsymbol{E}\cdot\boldsymbol{B}\,\mathrm{d}^3x$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\boldsymbol{B}_a, \boldsymbol{P}_b) = -2c\int_{\mathscr{V}_a} \boldsymbol{E}\cdot\boldsymbol{B}_a\,\mathrm{d}^3x - 2c\int_{\mathscr{V}_b} \boldsymbol{E}\cdot\boldsymbol{P}_b\,\mathrm{d}^3x. \tag{55}$$

In $\mathscr{V}_b$ it will be convenient to employ the gauge $\boldsymbol{P}_b = \nabla\times\boldsymbol{A}_{\mathrm{P}}$. In this case

$$\begin{aligned}\boldsymbol{E}\cdot\boldsymbol{P}_b &= \boldsymbol{E}\cdot\nabla\times\boldsymbol{A}_{\mathrm{P}}\\ &= \boldsymbol{A}_{\mathrm{P}}\cdot\nabla\times\boldsymbol{E} + \nabla\cdot(\boldsymbol{A}_{\mathrm{P}}\times\boldsymbol{E})\\ &= -\boldsymbol{A}_{\mathrm{P}}\cdot\frac{1}{c}\frac{\partial\boldsymbol{P}_b}{\partial t} + \nabla\cdot(\boldsymbol{A}_{\mathrm{P}}\times\boldsymbol{E}).\end{aligned} \tag{56}$$

Also,

$$\begin{aligned}\boldsymbol{A}_{\mathrm{P}}\cdot\frac{\partial\boldsymbol{P}_b}{\partial t} &= \boldsymbol{A}_{\mathrm{P}}\cdot\nabla\frac{\partial\psi_b}{\partial t}\\ &= \nabla\cdot\left(\frac{\partial\psi_b}{\partial t}\boldsymbol{A}_{\mathrm{P}}\right).\end{aligned} \tag{57}$$

Thus the $\mathscr{V}_b$ contribution to (55) becomes a surface integral:

$$-2c\int_{\mathscr{V}_b} \boldsymbol{E}\cdot\boldsymbol{P}_b\,\mathrm{d}^3x = 2c\oint_S \left(\boldsymbol{A}_{\mathrm{P}}\times\boldsymbol{E} - \frac{1}{c}\frac{\partial\psi_b}{\partial t}\boldsymbol{A}_{\mathrm{P}}\right)\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S. \tag{58}$$

Next consider the second term in (54):

$$\frac{\mathrm{d}}{\mathrm{d}t}H(\boldsymbol{P}_a,\boldsymbol{P}_b)=\int_{\mathscr{V}}\left(\boldsymbol{A}_{\mathrm{P}}\cdot\frac{\partial\boldsymbol{P}}{\partial t}+\boldsymbol{P}\cdot\frac{\partial\boldsymbol{A}_{\mathrm{P}}}{\partial t}\right)\mathrm{d}^3x, \tag{59}$$

$$=2\int_{\mathscr{V}}\boldsymbol{A}_{\mathrm{P}}\cdot\frac{\partial\boldsymbol{P}}{\partial t}\mathrm{d}^3x, \tag{60}$$

$$=2\oint_S\left(\frac{\partial\psi_a}{\partial t}-\frac{\partial\psi_b}{\partial t}\right)\boldsymbol{A}_{\mathrm{P}}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S. \tag{61}$$

Equations (54), (55), (58) and (61) combine to give

$$\frac{\mathrm{d}}{\mathrm{d}t}H_{\mathrm{R}}(\mathscr{V}_a)=-2c\int_{\mathscr{V}_a}\boldsymbol{E}\cdot\boldsymbol{B}_a\,\mathrm{d}^3x+2c\oint_S\left(\boldsymbol{A}_{\mathrm{P}}\times\boldsymbol{E}-\frac{1}{c}\frac{\partial\psi_a}{\partial t}\boldsymbol{A}_{\mathrm{P}}\right)\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S. \tag{62}$$

The volume integral gives the internal helicity dissipation, while the surface integral measures the flow of helicity across S. As $\boldsymbol{A}_{\mathrm{P}}$ and ψ_a are completely determined by $\boldsymbol{B}\cdot\hat{\boldsymbol{n}}|_S$, this equation is gauge-invariant. Note that the second term in the surface integral vanishes when S is a plane or a sphere, because then $\boldsymbol{A}_{\mathrm{P}}\cdot\hat{\boldsymbol{n}}|_S=0$ (50).

Let us compute the flow of relative helicity across a plane boundary for torsional motion on the plane and for circularly polarized Alfvén waves. Let S be the (x,y)-plane and $\mathscr{V}_a$ the upper half-space $z>0$. Assume that $\boldsymbol{B}\cdot\hat{\boldsymbol{n}}|_S$ is cylindrically symmetric, $\boldsymbol{E}=\boldsymbol{B}\times\boldsymbol{V}/c$, and employ coordinates (ρ,θ,z), where $\hat{\boldsymbol{n}}=-\hat{\boldsymbol{z}}$. From Stokes' theorem

$$\boldsymbol{A}_{\mathrm{P}}(\rho)=\hat{\boldsymbol{\theta}}\frac{\Phi(\rho)}{2\pi\rho}, \tag{63}$$

where $\Phi(\rho)=2\pi\int_0^\rho B_z\rho'\,\mathrm{d}\rho'$ is the flux contained within radius ρ. For torsional motions assume

$$\boldsymbol{V}=\rho\omega(\rho)\,\hat{\boldsymbol{\theta}}. \tag{64}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}H_{\mathrm{R}}(\mathscr{V}_a)=2c\int_S\boldsymbol{A}_{\mathrm{P}}\times\boldsymbol{E}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S, \tag{65}$$

$$=-2\int_S(\boldsymbol{A}_{\mathrm{P}}\cdot\boldsymbol{V})\,B_z\,\mathrm{d}S. \tag{66}$$

Thus, the flow into $\mathscr{V}_a$ is, from (63) and (64),

$$\frac{\mathrm{d}}{\mathrm{d}t}H_{\mathrm{R}}(\mathscr{V}_a)=-2\int\frac{\omega(\rho)}{2\pi}\,\Phi(\rho)\,\mathrm{d}\Phi(\rho). \tag{67}$$

This result is what one would expect from the analysis of twist helicity given in §2. The annulus with flux $\mathrm{d}\Phi(\rho)$ wraps around the flux inside $\omega(\rho)/2\pi$ times per second (compare with (12)). Note that right-handed vorticity twists the magnetic field above S with negative helicity.

The analysis of circularly polarized waves is somewhat more complicated. Let $\boldsymbol{B}_0=B_0\hat{\boldsymbol{z}}$ be uniform on S, and let the fluctuation fields be

$$\boldsymbol{B}_1=(\hat{\boldsymbol{x}}+\mathrm{i}\hat{\boldsymbol{y}})\,B_1\,\mathrm{e}^{\mathrm{i}(kz-\omega t)}; \tag{68}$$

$$\boldsymbol{V}_1=-\frac{V_{\mathrm{A}}}{B_0}\boldsymbol{B}_1. \tag{69}$$

Here V_{A} is the Alfvén velocity of the wave. For these fields

$$-2\int_S\mathrm{d}S\,(\boldsymbol{A}_{\mathrm{P}}\cdot\boldsymbol{V}_1)\,\boldsymbol{B}\cdot\hat{\boldsymbol{n}}=0, \tag{70}$$

when integrated over either θ or over one period (as $\boldsymbol{A}_{\mathrm{P}}$ is independent of time). This result must be regarded with caution because we have let $\boldsymbol{B}_1$ be constant as $\rho \rightarrow \infty$. Thus neither gauge invariance of the relative-helicity formalism nor special linkage properties at infinity can be assumed. One may note that the field lines of $\boldsymbol{B}_0+\boldsymbol{B}_1$, which individually have helical shape, nevertheless do not wrap about each other. As the infinitely extending waves resemble ripples in unlinked field lines, it is reasonable to find zero net topological flow in the above calculation.

The situation changes when we consider circularly polarized *wave packets*, finite in the $\hat{\boldsymbol{x}}$- and $\hat{\boldsymbol{y}}$-directions. Here $B_1 = B_1(x,y)$. The wave packets can be approximated by the divergence-free field

$$\boldsymbol{B}_1 = \frac{1}{2\pi}\int \mathrm{d}^2q\, b(\boldsymbol{q})\, \mathrm{e}^{\mathrm{i}(q_x x+q_y y)}\left[(\hat{\boldsymbol{x}}+\mathrm{i}\hat{\boldsymbol{y}})-\frac{1}{k}(q_z+\mathrm{i}q_y)\,\hat{\boldsymbol{z}}\right]\mathrm{e}^{\mathrm{i}(kz-\omega t)}, \tag{71}$$

where $b(\boldsymbol{q})$ is the two-dimensional Fourier transform of $B_1(x,y)$. The condition $\nabla\cdot\boldsymbol{B}_1 = 0$ requires the addition of a z-component to the field. Note that $\boldsymbol{B}_1$ satisfies the condition $\nabla\times\boldsymbol{B}_1 = k\boldsymbol{B}_1$ to first order in the ratio of wavelength to horizontal scale length of $\boldsymbol{B}_1$. We will assume that this ratio is very small, i.e. that $b(\boldsymbol{q})$ is substantial only for $q \ll k$. The z-component of $\boldsymbol{B}$ allows for field-line connections at the boundaries of the wave packet, which were not manifest in the plane-wave calculation.

Because the boundary function $\boldsymbol{B}_1\cdot\hat{\boldsymbol{n}}|_{z=0}$ now has a z-component, $\boldsymbol{A}_{\mathrm{P}}$ will contain an oscillating part $\boldsymbol{A}_{\mathrm{P1}}$. To obtain $\boldsymbol{A}_{\mathrm{P1}}$ at $z=0$, note that $\boldsymbol{A}_{\mathrm{P1}}\cdot\hat{\boldsymbol{z}} = 0$, and that

$$\frac{\partial A_{\mathrm{P1}x}}{\partial x}+\frac{\partial A_{\mathrm{P1}y}}{\partial y} = 0, \tag{72}$$

$$\frac{\partial A_{\mathrm{P1}y}}{\partial x}-\frac{\partial A_{\mathrm{P1}x}}{\partial y} = B_{1z}. \tag{73}$$

The result is

$$\boldsymbol{A}_{\mathrm{P1}} = \frac{1}{2\pi}\int \mathrm{d}^2q\, b(\boldsymbol{q})\, \mathrm{e}^{\mathrm{i}(q_z x+q_y y)}\,\frac{q_x+\mathrm{i}q_y}{\mathrm{i}kq^2}\,[q_y\hat{\boldsymbol{x}}-q_x\hat{\boldsymbol{y}}]\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{74}$$

We can now compute the time average of $(\mathrm{d}/\mathrm{d}t)\,H_{\mathrm{R}}(\mathscr{V}_a)$: from (65)

$$\left\langle\frac{\mathrm{d}}{\mathrm{d}t}H_{\mathrm{R}}(\mathscr{V}_a)\right\rangle = \mathrm{Re}\int_{z=0}\mathrm{d}S\,c\boldsymbol{E}\times\boldsymbol{A}^*_{\mathrm{P1}}\cdot\hat{\boldsymbol{z}}. \tag{75}$$

Here $\boldsymbol{E} = \boldsymbol{B}_0\times \boldsymbol{V}_1/c$. After some algebra,

$$\left\langle\frac{\mathrm{d}}{\mathrm{d}t}H_{\mathrm{R}}(\mathscr{V}_a)\right\rangle = \frac{V_{\mathrm{A}}}{k}\int |b(\boldsymbol{q})|^2\,\mathrm{d}^2q, \tag{76}$$

$$= \frac{V_{\mathrm{A}}}{k}\int |B_1(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y. \tag{77}$$

An analogous result is obtained for circularly polarized electromagnetic waves moving at the velocity c. These results (for finite wave packets) are consistent with the interpretation of circularly polarized waves as carrying a helicity density $\langle h\rangle = B_1^2/k$ (Montgomery & Turner 1981; Matthaeus & Smith 1981).

5. Conclusions

Magnetic helicity has been shown to be closely associated with many aspects of the topological structure of the field. In §2 H has been classified into internal and external helicity, and into contributions from twist number and writhing number.

The internal helicity arises from kinks and twist within a flux tube, whereas external helicity arises from knotting and linkage. Internal and external helicity are separately conserved in ideal MHD. In plasmas with high but finite magnetic Reynolds number, it has been conjectured (Taylor 1974) that reconnection of field lines can alter the field topology while approximately conserving H. In this case, helicity can be transferred between external and internal sources. For example, two linked flux tubes with external helicity $2\Phi^2$ could reconnect into one tube with zero external helicity, but containing two extra units of internal helicity. Thus reconnection can be a source of twist for flux tubes (Berger 1982).

Helicity was also related to the twist number and writhing number of the field lines. The twist number measures the torsion of a line about the central axis of a flux tube, while the writhing number arises from the twisting, knotting, and linking of the central axes of flux tubes.

Magnetic helicity is only gauge-invariant and topologically well-defined when integrated over a volume bounded by magnetic surfaces. However, a relative measure of the helicity of a simply connected volume with open field lines does exist. As shown in §3, the relative contributions to the helicity of all space of two configurations with common extensions outside the volume does not depend upon details of the extension. A measure of the topological content of a volume can thus be obtained by comparing a given field with the corresponding potential field. This measure has been called relative helicity. Potential fields have a minimum of structure for the given boundary conditions. For example, if the boundary surface is a plane or a sphere, and potential fields are placed on either side of the boundary, the total helicity is zero. It is interesting to note that, for such boundaries, the sum of the relative helicities of the fields residing in the two subvolumes of space equals the helicity of the total field.

The time derivative of relative helicity has a simple form (62): a volume integral of $\boldsymbol{E}\cdot\boldsymbol{B}$, plus a surface integral. The $\boldsymbol{E}\cdot\boldsymbol{B}$ term represents dissipation in plasmas with finite resistivity. In high-magnetic-Reynolds-number plasmas, however, this term can often be considered negligible (Taylor 1974, 1981; Berger 1984). The surface term provides a well-defined measure for the flow of topological structure into a subvolume of space. This measure has been shown to provide physically reasonable results for torsional motions on the boundary plane and for circularly polarized waves.

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